

SEGAL FRÉCHET ALGEBRAS

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ABSTRACT. Let $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$ be a Fréchet algebra. In this paper, we introduce the concept of Segal Fréchet algebra and investigate known results about abstract Segal algebras, for Segal Fréchet algebras. Also we recall the concept of approximate identities for topological algebras and provide some remarkable results for Segal Fréchet algebras. Moreover, we verify ideal theorem for Fréchet algebras and characterize closed ideals of Segal Fréchet algebra $(\mathcal{B}, q_m)_{m \in \mathbb{N}}$ in $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$.

0. Introduction

Segal algebras were first defined by H. Reiter for group algebras in [20]. Then Burnham [2] introduced the notion of abstract Segal algebra in Banach algebra $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$. As an important result, he showed that approximate identities in the proper abstract Segal algebras can not be bounded. Also he examined ideal structure of a class of Banach algebras that subsume Cigler's normed ideals [6] and Reiter's Segal algebras [19, Page 127]. Moreover, he proved that if $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a commutative abstract Segal algebra in $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ with an approximate identity, then there is a one to one correspondence between the closed ideals of \mathcal{B} and those of \mathcal{A} . We also found many other valuable results in [1]. In fact he considered the relationship between the Banach algebras $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ and $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$, whenever \mathcal{B} is also a left ideal in \mathcal{A} . For example, he proved that if $\|b\|_{\mathcal{A}} \leq D\|b\|_{\mathcal{B}}$, for all $b \in \mathcal{B}$ and some $D > 0$, then \mathcal{B} is a Banach left \mathcal{A} -module [1, Theorem 2.3]. Moreover, he showed that if there is the certain above inequality between $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$, and also \mathcal{B} is a Banach left \mathcal{A} -module, then \mathcal{B} is a left ideal in $cl_{\mathcal{A}}(\mathcal{B})$, closure \mathcal{B} in \mathcal{A} [1, Proposition 2.1]. In fact these observations lead to summarize the definition of abstract Segal algebras. We also refer to [3], [4], [6], [7], [8], [10], [11] and [21], which contain valuable results related to this subject.

Some of the notions related to Banach algebras, have been introduced and studied for Fréchet algebras. For example, the notion of amenability of a Fréchet algebra was introduced by A. Yu. Pirkovskii [18]. He generalized some theorems about amenability of Banach algebras such as strictly flat Banach A -bimodule, virtual diagonal and approximate diagonal of Banach algebras, to Fréchet algebras. Also in [16], P. Lawson and C. J. Read introduced and studied some notions about approximate amenability and approximate contractibility of Fréchet algebras.

The present work is essentially raised from the available results in the field of abstract Segal algebras. Let $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$ be a Fréchet algebra. According to the definition of abstract Segal algebras [2], we first introduce the concept of Segal Fréchet algebra in $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$. To provide an example about Segal Fréchet algebra, we verify the results of Burnham and Barnes for Segal Fréchet algebras

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and show that Proposition 2.1 and Theorem 2.3 of [1] are valid for Segal Fréchet algebras, as well. It follows that as in abstract Segal algebras, the definition of Segal Fréchet algebra can be summarized. Moreover, we recall the concept of approximate units, multiple approximate identity and approximate identity for Fréchet algebras and investigate [2, Theorem 1.2] for Segal Fréchet algebras and obtain the same result. Indeed, we prove that if $(\mathcal{B}, q_m)_{m \in \mathbb{N}}$ is a proper Segal Fréchet algebra in $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$ which contains an approximate identity (e_α) , then (e_α) can not be bounded in $(\mathcal{B}, q_m)_{m \in \mathbb{N}}$. At the end, we verify ideal theorem and characterize closed left ideals of a Segal Fréchet algebra. In fact we show that every closed left ideal of any Segal Fréchet algebra in $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$, is the intersection of a closed left ideal of \mathcal{A} with \mathcal{B} .

1. Preliminaries

In this section, we present some basic definitions related to Fréchet algebras, which will be required throughout the paper. See [12], [13] and [17], for more information.

A locally convex topological vector space E is a topological vector space in which the origin has a local base of absolutely convex absorbent sets. A collection \mathcal{U} of zero neighborhoods in E is called a fundamental system of zero neighborhoods, if for every zero neighborhood U , there exists a $V \in \mathcal{U}$ and an $\varepsilon > 0$ such that $\varepsilon V \subset U$. Throughout the paper, all locally convex spaces are assumed to be Hausdorff. $S \subseteq E$ is called bounded if for every zero neighborhood U , there exists scalar λ such that $S \subseteq \lambda U$; it is called balanced if for each $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $\alpha S \subseteq S$. Moreover S is called absorbing if for each $x \in E$, there is the scalar λ such that $x \in \lambda S$.

A family $(p_\alpha)_{\alpha \in A}$ of continuous seminorms on E is called a fundamental system of seminorms, if the sets

$$U_\alpha = \{x \in E : p_\alpha(x) < 1\} \quad (\alpha \in A)$$

form a fundamental system of zero neighborhoods. We refer to [17, page 251], for more details. Every Hausdorff locally convex space E has a fundamental system of seminorms $(p_\alpha)_{\alpha \in A}$; equivalently a family of the seminorms satisfying the following properties:

- (i) For every $x \in E$ with $x \neq 0$, there exists an $\alpha \in A$ with $p_\alpha(x) > 0$;
- (ii) For all $\alpha, \beta \in A$, there exist $\gamma \in A$ and $C > 0$ such that

$$\max(p_\alpha(x), p_\beta(x)) \leq C p_\gamma(x) \quad (x \in E);$$

see [17, Lemmas 22.4, 22.5].

Now let E be a locally convex space and $(p_\alpha)_{\alpha \in \Lambda}$ be a fundamental system of seminorms. A subset B of E is bounded if and only if $\sup_{x \in B} p_\alpha(x) < \infty$, for each $\alpha \in \Lambda$.

We recall [17, Proposition 22.6], which is very useful in our later discussions.

Proposition 1.1. *Let E and F be locally convex spaces with the fundamental system of seminorms $(p_\alpha)_{\alpha \in A}$ in E and $(q_\beta)_{\beta \in B}$ in F . Then for every linear mapping $T : E \rightarrow F$, the following assertions are equivalent.*

- (i) T is continuous.
- (ii) T is continuous at 0.
- (iii) For each $\beta \in B$ there exist an $\alpha \in A$ and $C > 0$, such that

$$q_\beta(T(x)) \leq C p_\alpha(x),$$

for all $x \in E$.

It should be noted that by [13, page 24], if (E, p_μ) , (F, q_λ) and (G, r_ν) are locally convex spaces, and $\theta : E \times F \rightarrow G$ is a bilinear map, then θ is jointly continuous if and only if for any ν_0 there exist μ_0 and λ_0 such that the bilinear map

$$\theta : (E, p_{\mu_0}) \times (F, q_{\lambda_0}) \longrightarrow (G, r_{\nu_0})$$

is jointly continuous. In other words there exists $C > 0$ such that

$$r_{\nu_0}(\theta(x, y)) \leq C p_{\mu_0}(x) q_{\lambda_0}(y),$$

for all $x \in E$ and $y \in F$. Recall from [22] that bilinear map f from $E \times F$ into G is said to be separately continuous if all partial maps $f_x : F \rightarrow G$ and $f_y : E \rightarrow G$ defined by $y \mapsto f(x, y)$ and $x \mapsto f(x, y)$, respectively, are continuous for each $x \in E$ and $y \in F$. By [22, chapter.III.5.1], both types of continuity coincide in the class of Fréchet spaces and in particular, Banach spaces. In such a situation, we use only "continuous" phrase. A topological algebra \mathcal{A} is an algebra, which is a topological vector space and the multiplication $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$, defined by $(a, b) \mapsto ab$ is a jointly continuous mapping; see [12, Definition (3.1.5)]. A locally m -convex Fréchet algebra (lmc Fréchet algebra) is a complete topological algebra, whose topology is given by a countable family of increasing submultiplicative seminorms; see [12] and [14] for more information. For convenience, throughout the paper, we use the label "Fréchet algebra", instead of "lmc Fréchet algebra".

2. Introduction of Segal Fréchet algebra

A Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is called an abstract Segal algebra of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ if the following statements are satisfied:

- (A₁) \mathcal{B} is a dense left ideal in \mathcal{A} .
- (A₂) There exists $M > 0$ such that $\|f\|_{\mathcal{A}} \leq M \|f\|_{\mathcal{B}}$, for each $f \in \mathcal{B}$.
- (A₃) There exists $C > 0$ such that $\|fg\|_{\mathcal{B}} \leq C \|f\|_{\mathcal{A}} \|g\|_{\mathcal{B}}$, for each $f, g \in \mathcal{B}$.

Equivalently, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is an abstract Segal algebra of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ if it is continuously embedded in \mathcal{A} and also it is a Banach left \mathcal{A} -module. Retrieved from this definition, we introduce the concept of Segal Fréchet algebra as the following.

Definition 2.1. A Fréchet algebra $(\mathcal{B}, q_m)_{m \in \mathbb{N}}$ is a Segal Fréchet algebra in a Fréchet algebra $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$ if the following conditions are satisfied:

- (i) \mathcal{B} is a dense left ideal in \mathcal{A} .
- (ii) The map

$$(2.1) \quad i : (\mathcal{B}, q_m)_{m \in \mathbb{N}} \longrightarrow (\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}, \quad a \mapsto a, \quad (a \in \mathcal{B})$$

is continuous.

- (iii) The map

$$(2.2) \quad (\mathcal{B}, p_\ell)_{\ell \in \mathbb{N}} \times (\mathcal{B}, q_m)_{m \in \mathbb{N}} \longrightarrow (\mathcal{B}, q_m)_{m \in \mathbb{N}}, \quad (a, b) \mapsto ab, \quad (a, b \in \mathcal{B})$$

is jointly continuous.

\mathcal{B} is called a symmetric Segal Fréchet algebra in \mathcal{A} , if \mathcal{B} is a dense two-sided ideal in \mathcal{A} and (2.1) and (2.2) hold. Moreover the map

$$(\mathcal{B}, q_m)_{m \in \mathbb{N}} \times (\mathcal{B}, p_\ell)_{\ell \in \mathbb{N}} \longrightarrow (\mathcal{B}, q_m)_{m \in \mathbb{N}}, \quad (a, b) \mapsto ab, \quad (a, b \in \mathcal{B})$$

is jointly continuous.

Note that the concept of Segal Fréchet algebra is coincided to the concept of abstract Segal algebras, in the case where \mathcal{A} and \mathcal{B} are Banach algebras.

For presenting an example about Definition 2.1, we shall make some preparations.

Remark 2.2. Let $(\mathcal{B}, q_m)_{m \in \mathbb{N}}$ be a Segal Fréchet algebra in Fréchet algebra $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$.

- (1) As in Banach algebras, conditions (ii) and (iii) in Definition 2.1, can be given similar to (A_2) and (A_3) in the definition of abstract Segal algebras. In fact Proposition 1.1 implies that condition (ii) in Definition 2.1 is equivalent to the fact that, for every $\ell \in \mathbb{N}$, there exist $M_\ell > 0$ and $m_\ell \in \mathbb{N}$ such that, $p_\ell(b) \leq M_\ell q_{m_\ell}(b)$, for all $b \in \mathcal{B}$. Moreover by [13, Page 24], continuity of the map in (iii) is equivalent to the fact that for every $m \in \mathbb{N}$ there exist $K_m > 0$ and $\ell_m, n_m \in \mathbb{N}$ such that for all $a, b \in \mathcal{B}$,

$$q_m(ab) \leq K_m p_{\ell_m}(a) q_{n_m}(b).$$

- (2) As in abstract Segal algebras, since \mathcal{B} is a dense subspace of \mathcal{A} , we can extend the continuous map (2.2) to the unique continuous map

$$(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}} \times (\mathcal{B}, q_m)_{m \in \mathbb{N}} \longrightarrow (\mathcal{B}, q_m)_{m \in \mathbb{N}}, \quad (a, b) \mapsto ab.$$

Indeed, since

$$(\mathcal{B}, p_\ell)_{\ell \in \mathbb{N}} \times (\mathcal{B}, q_m)_{m \in \mathbb{N}} \longrightarrow (\mathcal{B}, q_m)_{m \in \mathbb{N}}, \quad (a, b) \mapsto ab,$$

is jointly continuous, so it is separately continuous and consequently both maps

$$(\mathcal{B}, p_\ell)_{\ell \in \mathbb{N}} \longrightarrow (\mathcal{B}, q_m)_{m \in \mathbb{N}}, \quad a \mapsto ab,$$

and

$$(\mathcal{B}, q_m)_{m \in \mathbb{N}} \longrightarrow (\mathcal{B}, q_m)_{m \in \mathbb{N}}, \quad b \mapsto ab,$$

are continuous, for all $a, b \in \mathcal{B}$. It is not hard to see that for each $a \in \mathcal{A}$, the map

$$(\mathcal{B}, q_m)_{m \in \mathbb{N}} \longrightarrow (\mathcal{B}, q_m)_{m \in \mathbb{N}}, \quad b \mapsto ab,$$

is continuous. Moreover by [17, Lemma 22.19], the map

$$(\mathcal{B}, p_\ell)_{\ell \in \mathbb{N}} \longrightarrow (\mathcal{B}, q_m)_{m \in \mathbb{N}}, \quad a \mapsto ab,$$

has a unique extension to the continuous map

$$(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}} \longrightarrow (\mathcal{B}, q_m)_{m \in \mathbb{N}}, \quad a \mapsto ab,$$

for all $b \in \mathcal{B}$. It follows that the map

$$(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}} \times (\mathcal{B}, q_m)_{m \in \mathbb{N}} \longrightarrow (\mathcal{B}, q_m)_{m \in \mathbb{N}},$$

is separately continuous. Since \mathcal{A} and \mathcal{B} are Fréchet algebra, thus the map is jointly continuous.

As the main results of this section, we prove Proposition 2.1 and Theorem 2.3 of [1], for the Fréchet algebras. In fact we show that the definition of Segal Fréchet algebra can be summarized. First, we recall closed graph theorem for the Fréchet spaces. Let E and F be Fréchet spaces and $T : E \longrightarrow F$ is a linear mapping such that its graph, $\{(x, T(x)) : x \in E\}$, is closed in $E \times F$. Then T is continuous; see [12, B.2].

Theorem 2.3. *Let $(\mathcal{B}, q_m)_{m \in \mathbb{N}}$ be a Fréchet algebra, which is a left ideal in a Fréchet algebra $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$. If the map*

$$i : (\mathcal{B}, q_m)_{m \in \mathbb{N}} \longrightarrow (\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}} \quad a \mapsto a$$

is continuous, then the map

$$(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}} \times (\mathcal{B}, q_m)_{m \in \mathbb{N}} \longrightarrow (\mathcal{B}, q_m)_{m \in \mathbb{N}}, \quad (a, b) \mapsto ab$$

is continuous.

Proof. Let $b \in \mathcal{B}$ and define

$$T_b : (\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}} \longrightarrow (\mathcal{B}, q_m)_{m \in \mathbb{N}}$$

by $T_b(a) = ab$, for each $a \in \mathcal{A}$. By the closed graph theorem in Fréchet algebras, we show that T_b is continuous. Suppose that $(a_n)_{n \in \mathbb{N}}$ is a sequence in \mathcal{A} such that $\lim_{n \rightarrow \infty} p_\ell(a_n) = 0$ and

$$(2.3) \quad \lim_{n \rightarrow \infty} q_m(T_b(a_n) - c) = 0,$$

for all $\ell, m \in \mathbb{N}$ and some $c \in \mathcal{B}$. We show that $c = 0$. By the hypothesis for each $\ell \in \mathbb{N}$, there exist $M_\ell > 0$ and $m_\ell \in \mathbb{N}$, such that

$$p_\ell(a) \leq M_\ell q_{m_\ell}(a), \quad (a \in \mathcal{B})$$

and so for each $n \in \mathbb{N}$

$$(2.4) \quad p_\ell(a_n b - c) \leq M_\ell q_{m_\ell}(a_n b - c).$$

By (2.3), the right hand side of the above inequality tends to zero. The inequality (2.4) implies that $\lim_{n \rightarrow \infty} a_n b = c$, in the topology of \mathcal{A} . Moreover, since p_ℓ is submultiplicative, we have $p_\ell(a_n b) \leq p_\ell(a_n) p_\ell(b)$. It follows that $\lim_{n \rightarrow \infty} a_n b = 0$, in the topology of \mathcal{A} . Thus $c = 0$, as claimed. Consequently T_b is continuous. Similarly, one can show that for each $a \in \mathcal{A}$, the map $S_a : (\mathcal{B}, q_m)_{m \in \mathbb{N}} \rightarrow (\mathcal{B}, q_m)_{m \in \mathbb{N}}$ defined by $S_a(b) = ab$ is continuous. This completes the proof. \square

Theorem 2.4. *Let $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$ and $(\mathcal{B}, q_m)_{m \in \mathbb{N}}$ be Fréchet algebras, such that \mathcal{B} is a subalgebra of \mathcal{A} and both maps given in (ii) and (iii) in Definition 2.1 are continuous. Then \mathcal{B} is a left ideal in $cl_{\mathcal{A}}(\mathcal{B})$, the closure of \mathcal{B} in \mathcal{A} .*

Proof. Suppose that $b \in \mathcal{B}$ and $a \in cl_{\mathcal{A}}(\mathcal{B})$. Thus there exists a sequence $(a_n)_{n \in \mathbb{N}}$ in \mathcal{B} such that $\lim_{n \rightarrow \infty} p_\ell(a_n - a) = 0$, for every $\ell \in \mathbb{N}$. Also by Remark 2.2, for each $m \in \mathbb{N}$, there exist $C_m > 0$ and $n_m, \ell_m \in \mathbb{N}$ such that

$$q_m(a_n b - a_k b) = q_m((a_n - a_k)b) \leq C_m p_{\ell_m}(a_n - a_k) q_{n_m}(b),$$

for all $n, k \in \mathbb{N}$. Since $(a_n)_{n \in \mathbb{N}}$ is convergent in \mathcal{A} , it is a Cauchy sequence in \mathcal{A} , and so $(a_n b)_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{B} . Consequently there exists $c \in \mathcal{B}$ such that for every $m \in \mathbb{N}$, $\lim_{n \rightarrow \infty} q_m(a_n b - c) = 0$. On the other hand by Remark 2.2, for every $\ell \in \mathbb{N}$, there exist $C_\ell > 0$ and $m_\ell \in \mathbb{N}$ such that

$$p_\ell(a_n b - c) \leq C_\ell q_{m_\ell}(a_n b - c).$$

Since the right hand side of the above inequality tends to zero, $\lim_{n \rightarrow \infty} a_n b = c$, in the topology of \mathcal{A} . Also by the continuity of multiplication in \mathcal{A} , $\lim_{n \rightarrow \infty} a_n b = ab$. It follows that $ab = c$, and so $ab \in \mathcal{B}$. This completes the proof. \square

Corollary 2.5. *Let $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$ and $(\mathcal{B}, q_m)_{m \in \mathbb{N}}$ be Fréchet algebras, such that \mathcal{B} is a dense subalgebra of \mathcal{A} and both maps given in (ii) and (iii) in Definition 2.1 are continuous. Then \mathcal{B} is a Segal Fréchet algebra.*

Remark 2.6. Let $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$ and $(\mathcal{B}, q_m)_{m \in \mathbb{N}}$ be Fréchet algebras. The definition of Segal Fréchet algebra may be summarized as the following:

- (1) By Theorem 2.3, part (iii) in Definition 2.1 can be omitted. In fact a Fréchet algebra $(\mathcal{B}, q_m)_{m \in \mathbb{N}}$ is a Segal Fréchet algebra in Fréchet algebra $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$, if the conditions (i) and (ii) in Definition 2.1 are satisfied.
- (2) If \mathcal{B} is a dense subalgebra of \mathcal{A} , then Corollary 2.5 mentions that in Definition 2.1, condition (i) can be obtained by (ii) and (iii).

Now we are in a position to provide examples concerning Segal Fréchet algebras. We found many examples in [23], which satisfy the conditions of Segal Fréchet algebras and so can be good examples of our definition.

We explain here Part (b) of [23, Example 3.3], which is a nice example in this field.

Example 2.7. Let X be a infinite countable set. A function $\ell : X \rightarrow [1, \infty)$ is a scale on X . We say that a scale ℓ on X is proper if the inverse map ℓ^{-1} takes bounded subsets of $[1, \infty)$ to finite subsets of X . For the family of scales $\ell = \{\ell^n\}_{n=0}^\infty$ on X , define

$$\mathcal{S}_\ell^\infty(X) = \{\varphi : X \rightarrow \mathbb{C}, \|\varphi\|_n^\infty < \infty, \forall n \in \mathbb{N}\},$$

where

$$\|\varphi\|_n^\infty = \sup_{x \in X} \{\ell^n(x) |\varphi(x)|\}.$$

Then $\mathcal{S}_\ell^\infty(X)$ is called the sup-norm ℓ -rapidly vanishing functions on X . The family $(\ell^n)_{n=0}^\infty$ will satisfy $\ell^0 \leq \ell^1 \leq \dots \leq \ell^n \leq \dots$, so that the families of norms $\{\|\cdot\|_n^\infty\}_{n=0}^\infty$ are increasing. Moreover, it is easy to see that all of them are submultiplicative under pointwise product. In fact $\mathcal{S}_\ell^\infty(X)$ is a Fréchet algebra. Now consider $c_0(X)$, the commutative Banach algebra of complex-valued sequences which vanish at infinity, with pointwise multiplication and sup-norm $\|\cdot\|_\infty$. We show that $\mathcal{S}_\ell^\infty(X) \subseteq c_0(X)$. Suppose on the contrary that there exists $\varphi \in \mathcal{S}_\ell^\infty(X)$ such that $\varphi \notin c_0(X)$. Thus there is $\varepsilon > 0$ such that for each finite subset F of X , there exists $x_F \notin F$ with $|\varphi(x_F)| \geq \varepsilon$. Since ℓ is a proper scale, thus for each $n \in \mathbb{N}$, there exists $x_n \notin \ell^{-1}([1, n])$ such that $|\varphi(x_n)| \geq \varepsilon$. It follows that

$$\sup_{x \in X} \{\ell(x) |\varphi(x)|\} = \infty,$$

which contradicts the assumption of $\varphi \in \mathcal{S}_\ell^\infty(X)$. Therefore $\mathcal{S}_\ell^\infty(X) \subseteq c_0(X)$. It is easy to see that the inequalities $\|\varphi \psi\|_n^\infty \leq \|\varphi\|_n^\infty \|\psi\|_\infty$ are satisfied, for all $\varphi, \psi \in \mathcal{S}_\ell^\infty(X)$. Since $\mathcal{S}_\ell^\infty(X)$ contains the space of finite support functions denoted by $c_{00}(X)$, it follows that $\mathcal{S}_\ell^\infty(X)$ is a dense Fréchet ideal in $c_0(X)$, and so $\mathcal{S}_\ell^\infty(X)$ is a Segal Fréchet algebra in $c_0(X)$.

3. Main results

In this section, we prove some other results of [1] and [2] for Segal Fréchet algebras. We require recall the following definitions from [14] and [18].

Definition 3.1. Let $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$ be a Fréchet algebra.

- (1) We say that \mathcal{A} has left (right) approximate units if for each $x \in \mathcal{A}$ and $\ell \in \mathbb{N}$ and $\varepsilon > 0$, there exists $u \in \mathcal{A}$ such that $p_\ell(ux - x) < \varepsilon$ ($p_\ell(xu - x) < \varepsilon$). Moreover we say that \mathcal{A} has a bounded left (right) approximate units if there exists a bounded subset B of \mathcal{A} , such that for each $x \in \mathcal{A}$ and $\varepsilon > 0$ and $\ell \in \mathbb{N}$, there exists $b \in B$ such that

$$p_\ell(bx - x) < \varepsilon \quad (p_\ell(xb - x) < \varepsilon).$$

- (2) \mathcal{A} has a left (right) multiple approximate identity if given $\{a_1, \dots, a_n\} \subseteq \mathcal{A}$ and $\varepsilon > 0$ and $\ell \in \mathbb{N}$, we may find $b \in \mathcal{A}$ such that

$$p_\ell(ba_i - a_i) < \varepsilon \quad (p_\ell(a_i b - a_i) < \varepsilon),$$

for all $i = 1, 2, \dots, n$. We further say that \mathcal{A} has a bounded left (right) multiple approximate identity if there exists a bounded subset B of \mathcal{A} , such that for each finite subset $\{a_1, \dots, a_n\}$ of \mathcal{A} , we may find b in B , satisfying the above inequality.

- (3) A left (right) approximate identity in \mathcal{A} is a net $(e_\alpha)_{\alpha \in \Lambda}$ in \mathcal{A} , such that

$$\lim_{\alpha} p_\ell(e_\alpha a - a) = 0 \quad (\lim_{\alpha} p_\ell(ae_\alpha - a) = 0),$$

for all $a \in \mathcal{A}$ and $\ell \in \mathbb{N}$. It is called a bounded left (right) approximate identity if $\{e_\alpha, \alpha \in \Lambda\}$ is a bounded set in \mathcal{A} . We say that $(e_\alpha)_{\alpha \in \Lambda}$ is an approximate identity if it is both left and right approximate identity.

We commence with the following result, which is in fact a generalization of [5, Proposition 2 page 58], to Fréchet algebras.

Proposition 3.2. *Let $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$ be a Fréchet algebra. Then the following statements are equivalent;*

- (i) \mathcal{A} has bounded left approximate units,
- (ii) \mathcal{A} has a bounded left multiple approximate identity,
- (iii) \mathcal{A} has a bounded left approximate identity.

Proof. (i) \Rightarrow (ii). By the hypothesis, there exists a bounded subset B of \mathcal{A} such that for each $\ell \in \mathbb{N}$, $\sup_{b \in B} p_\ell(b) \leq M_\ell$, for some $M_\ell > 0$. Moreover, for each $x \in \mathcal{A}$ and $\varepsilon > 0$ and $\ell \in \mathbb{N}$, there exists $b \in B$ with $p_\ell(bx - x) < \varepsilon$. Set $W = \{u + v - vu : u, v \in B\}$. We show that for each finite subset F of \mathcal{A} and $\ell \in \mathbb{N}$ and $\varepsilon > 0$, there exists $w \in W$ such that for all $x \in F$, $p_\ell(wx - x) < \varepsilon$. We prove it inductively. First let $F = \{x_1, x_2\}$. So for each $\ell \in \mathbb{N}$, there exist $u, v \in B$ such that

$$p_\ell(x_1 - ux_1) < \frac{\varepsilon}{1 + M_\ell}$$

and

$$p_\ell((x_2 - ux_2) - v(x_2 - ux_2)) < \varepsilon.$$

Assuming $w = u + v - vu$, we have $p_\ell(x_j - wx_j) < \varepsilon$, for $j = 1, 2$. Now suppose that (ii) holds for $\{x_1, \dots, x_n\}$ and consider the finite subset $F = \{x_1, \dots, x_{n+1}\}$ of \mathcal{A} . Let $\alpha_\ell = \max\{p_\ell(x_j) : j = 1, \dots, n\}$. Thus for each $\ell \in \mathbb{N}$ there exists $y \in W$ such that

$$p_\ell(x_j - yx_j) < \frac{\varepsilon}{3(1 + M_\ell)^2}, \quad (j = 1, 2, \dots, n).$$

One can choose $w \in W$ such that

$$p_\ell(y - wy) < \frac{\varepsilon}{3\alpha_\ell} \quad \text{and} \quad p_\ell(x_{n+1} - wx_{n+1}) < \varepsilon.$$

Thus for each $j \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned} p_\ell(x_j - wx_j) &\leq p_\ell(x_j - yx_j) + p_\ell(yx_j - wyx_j) + p_\ell(wyx_j - wx_j) \\ &\leq p_\ell(x_j - yx_j) + p_\ell(y - wy)p_\ell(x_j) + p_\ell(w)p_\ell(yx_j - x_j) \\ &< \frac{\varepsilon}{3(1 + M_\ell)^2} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3(1 + M_\ell)^2}(2M_\ell + M_\ell^2) < \varepsilon, \end{aligned}$$

which is our claim.

(ii) \Rightarrow (i) is clear.

By [18, Proposition 5.2], (ii) and (iii), are equivalent. \square

The following result is interesting in its own right. It is a generalization of a result due to Burnham [2], for Segal Fréchet algebras. It shows that the condition (ii) in Definition 2.1, can be obtained from the conditions (i) and (iii).

Proposition 3.3. *Let (\mathcal{A}, p_ℓ) be a Fréchet algebra with a right approximate identity denoted by $(e_\alpha)_\alpha$, and (\mathcal{B}, q_m) be a Fréchet algebra, such that \mathcal{B} is a dense left ideal in (\mathcal{A}, p_ℓ) . Also suppose that the map*

$$(\mathcal{A}, p_\ell) \times (\mathcal{B}, q_m) \longrightarrow (\mathcal{B}, q_m), \quad (a, b) \mapsto ab, \quad (a \in \mathcal{A}, b \in \mathcal{B})$$

is continuous. Then (\mathcal{B}, q_m) is a Segal Fréchet algebra in (\mathcal{A}, p_ℓ) .

Proof. It is sufficient to show that the map (2.1) is continuous. We will use Closed graph theorem [17, Theorem 8.8] for Fréchet algebras. Let (b_n) be a sequence in \mathcal{B} such that $b_n \rightarrow 0$, in the topology of \mathcal{B} and $b_n \rightarrow c$, for some $c \in \mathcal{A}$, in the topology of \mathcal{A} . Since (\mathcal{B}, q_m) is a Fréchet algebra thus for every $b \in \mathcal{B}$, $b_n b \rightarrow 0$, in the topology of \mathcal{B} . By the hypothesis, $b_n b \rightarrow cb$, in the topology of \mathcal{B} . Thus $cb = 0$, for all $b \in \mathcal{B}$. Density of \mathcal{B} in \mathcal{A} implies that $ca = 0$, for all $a \in \mathcal{A}$. It follows that $ce_\alpha = 0$, for all α . Since $ce_\alpha \rightarrow_\alpha c$, in the topology of \mathcal{A} , therefore $c = 0$. Thus the result is obtained. \square

Corollary 3.4. *Let (\mathcal{A}, p_ℓ) be a Fréchet algebra with an approximate identity, and (\mathcal{B}, q_m) be a Fréchet algebra, such that \mathcal{B} is a dense two-sided ideal in (\mathcal{A}, p_ℓ) . Also suppose that the maps*

$$(\mathcal{A}, p_\ell) \times (\mathcal{B}, q_m) \longrightarrow (\mathcal{B}, q_m), \quad (a, b) \mapsto ab, \quad (a \in \mathcal{A}, b \in \mathcal{B})$$

and

$$(\mathcal{B}, q_m) \times (\mathcal{A}, p_\ell) \longrightarrow (\mathcal{B}, q_m), \quad (b, a) \mapsto ba, \quad (a \in \mathcal{A}, b \in \mathcal{B})$$

are continuous. Then (\mathcal{B}, q_m) is a symmetric Segal Fréchet algebra in (\mathcal{A}, p_ℓ) .

It is known that proper abstract Segal algebras never admit a bounded approximate identity. In the following result we obtain the same result for Segal Fréchet algebras.

Proposition 3.5. *Let $(\mathcal{B}, q_m)_{m \in \mathbb{N}}$ be a proper Segal Fréchet algebra in Fréchet algebra $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$, with an approximate identity $(e_\alpha)_{\alpha \in \Lambda}$. Then $(e_\alpha)_{\alpha \in \Lambda}$ is not bounded in \mathcal{B} .*

Proof. Suppose on the contrary that $(e_\alpha)_{\alpha \in \Lambda}$ is bounded in \mathcal{B} . Thus for each $m \in \mathbb{N}$, $\sup_{\alpha \in \Lambda} q_m(e_\alpha) < K_m$, for some $K_m > 0$. On the other hand by Remark 2.2, for every $m \in \mathbb{N}$, there exist $C_m > 0$ and $\ell_m, n_m \in \mathbb{N}$ such that

$$q_m(ab) \leq C_m p_{\ell_m}(a) q_{n_m}(b), \quad (a, b \in \mathcal{B}).$$

Since (e_α) is a bounded right approximate identity for \mathcal{B} , for every $b \in \mathcal{B}$, $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists $\alpha_m \in \Lambda$ such that for each $\alpha \geq \alpha_m$, we have

$$q_m(b) \leq q_m(be_\alpha) + \varepsilon \leq C_m p_{\ell_m}(b) q_{n_m}(e_\alpha) + \varepsilon \leq C_m K_{n_m} p_{\ell_m}(b) + \varepsilon.$$

It follows that for each $b \in \mathcal{B}$,

$$q_m(b) \leq C_m K_{n_m} p_{\ell_m}(b) + \varepsilon.$$

Since \mathcal{B} is dense in \mathcal{A} , it follows that $\mathcal{A} = \mathcal{B}$, which is a contradiction. \square

Proposition 3.6. *Let $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$ be a Fréchet algebra and \mathcal{B} be a dense subalgebra of \mathcal{A} . Moreover suppose that \mathcal{B} has a bounded left approximate identity $(e_\alpha)_{\alpha \in \Lambda}$. Then $(e_\alpha)_{\alpha \in \Lambda}$ is a bounded left approximate identity for \mathcal{A} .*

Proof. Since $(e_\alpha)_{\alpha \in \Lambda}$ is bounded in \mathcal{A} , we have $p_\ell(e_\alpha) \leq K_\ell$ for each $\ell \in \mathbb{N}$ and some $K_\ell > 0$. Clearly, we may assume that $K_\ell \geq 1$. Let $a \in \mathcal{A}$ and $\ell \in \mathbb{N}$. Since \mathcal{B} is dense in \mathcal{A} , for each $\varepsilon > 0$, there exists $b \in \mathcal{B}$ such that

$$p_\ell(b - a) < \frac{\varepsilon}{3K_\ell}.$$

On the other hand since $(e_\alpha)_{\alpha \in \Lambda}$ is a left approximate identity for \mathcal{B} , there exists $\alpha_\ell \in \Lambda$ such that for each $\alpha \geq \alpha_\ell$,

$$p_\ell(e_\alpha b - b) < \frac{\varepsilon}{3}.$$

Thus for each $\alpha \geq \alpha_\ell$ we have

$$\begin{aligned} p_\ell(e_\alpha a - a) &\leq p_\ell(e_\alpha a - e_\alpha b) + p_\ell(e_\alpha b - b) + p_\ell(b - a) \\ &\leq p_\ell(e_\alpha) p_\ell(a - b) + p_\ell(e_\alpha b - b) + p_\ell(b - a) \\ &\leq K_\ell p_\ell(a - b) + p_\ell(e_\alpha b - b) + p_\ell(b - a) \\ &< K_\ell \frac{\varepsilon}{3K_\ell} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3K_\ell} \leq \varepsilon. \end{aligned}$$

Consequently $(e_\alpha)_{\alpha \in \Lambda}$ is a left bounded approximate identity for \mathcal{A} . \square

The following corollary is immediately obtained by Proposition 3.6.

Corollary 3.7. *Let $(\mathcal{B}, q_m)_{m \in \mathbb{N}}$ be a Segal Fréchet algebra in Fréchet algebra $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$. Moreover suppose that \mathcal{B} has a left approximate identity $(e_\alpha)_{\alpha \in \Lambda}$ bounded in \mathcal{A} . Then $(e_\alpha)_{\alpha \in \Lambda}$ is a bounded left approximate identity for \mathcal{A} .*

Closed left (right) ideals in symmetric abstract Segal algebras have been characterized. In fact by ideal theorem, for every symmetric Segal algebra $S(G)$, every closed left (right) ideal I of $S(G)$ is of the form $J \cap S(G)$, where J is a unique closed left (right) ideal of $L^1(G)$. We refer to [9] and also [20] for the basic definition of Segal algebras and also all the required information about ideal theorem. Moreover this theorem has been proved for abstract Segal algebras; see [1, Theorem 3.1] and [1, Theorem 3.2]. As the final result of the present work, we prove this result for Segal Fréchet algebras.

Theorem 3.8. *Let $(\mathcal{B}, q_m)_{m \in \mathbb{N}}$ be a symmetric Segal Fréchet algebra in Fréchet algebra $(\mathcal{A}, p_\ell)_{\ell \in \mathbb{N}}$. Then the following statements hold.*

- (a) *If J is a left ideal in \mathcal{A} , then $cl_{\mathcal{A}}(J)$ (closure of J in \mathcal{A}) is a closed left ideal in \mathcal{A} .*
- (b) *If J is a left ideal in \mathcal{A} , then $cl_{\mathcal{A}}(J) \cap \mathcal{B}$ is a closed left ideal in \mathcal{B} .*
- (c) *If I is a left ideal in \mathcal{B} , then $cl_{\mathcal{A}}(I)$ is a closed left ideal in \mathcal{A} .*
- (d) *If I is a closed left ideal in \mathcal{B} and \mathcal{B} has left approximate units, then $I = cl_{\mathcal{A}}(I) \cap \mathcal{B}$.*

Proof. (a). Let $a \in \mathcal{A}$ and $b \in cl_{\mathcal{A}}(J)$. Then there exists a sequence $(b_n)_{n \in \mathbb{N}}$ in J such that $\lim_{n \rightarrow \infty} p_\ell(b_n - b) = 0$, for each $\ell \in \mathbb{N}$. Since p_ℓ is submultiplicative, thus

$$p_\ell(ab_n - ab) \leq p_\ell(a)p_\ell(b_n - b).$$

Thus $\lim_{n \rightarrow \infty} ab_n = ab$, in the topology of \mathcal{A} . Since $ab_n \in J$ for all $n \in \mathbb{N}$, it follows that $ab \in cl_{\mathcal{A}}(J)$. This gives the implication (a).

(b). By (a) it is clear that $cl_{\mathcal{A}}(J) \cap \mathcal{B}$ is a left ideal in \mathcal{B} . We prove that $cl_{\mathcal{A}}(J) \cap \mathcal{B}$ is closed. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $cl_{\mathcal{A}}(J) \cap \mathcal{B}$ and $a \in \mathcal{B}$ such that $\lim_{n \rightarrow \infty} a_n = a$, in the topology of \mathcal{B} . Thus $\lim_{n \rightarrow \infty} q_m(a_n - a) = 0$, for every $m \in \mathbb{N}$. It follows that $(a_n)_{n \in \mathbb{N}}$ tends to a , in the topology of \mathcal{A} . Since $cl_{\mathcal{A}}(J)$ is closed in \mathcal{A} , it follows that $a \in cl_{\mathcal{A}}(J)$. Consequently $cl_{\mathcal{A}}(J) \cap \mathcal{B}$ is a closed left ideal in \mathcal{B} .

(c). Let $a \in \mathcal{A}$ and $b \in cl_{\mathcal{A}}(I)$. Hence there exists a sequence $(b_n)_{n \in \mathbb{N}}$ in I such that $\lim_{n \rightarrow \infty} b_n = b$, in the topology of \mathcal{A} . So $\lim_{n \rightarrow \infty} p_{\ell}(b_n - b) = 0$, for each $\ell \in \mathbb{N}$. Since \mathcal{B} is dense in \mathcal{A} , there exists a sequence $(c_n)_{n \in \mathbb{N}}$ in \mathcal{B} such that $\lim_{n \rightarrow \infty} c_n = a$, in the topology of \mathcal{A} . Consequently $\lim_{n \rightarrow \infty} p_{\ell}(c_n - a) = 0$, for each $\ell \in \mathbb{N}$. Moreover we have

$$\begin{aligned} p_{\ell}(c_n b_n - ab) &\leq p_{\ell}((c_n - a)b_n) + p_{\ell}(a(b_n - b)) \\ &\leq p_{\ell}(c_n - a)p_{\ell}(b_n) + p_{\ell}(b_n - b)p_{\ell}(a). \end{aligned}$$

Since $(b_n)_{n \in \mathbb{N}}$ is a bounded sequence in \mathcal{A} , the right hand side of the above inequality tends to zero and so $(c_n b_n)_{n \in \mathbb{N}}$ tends to ab , in the topology of \mathcal{A} , which implies that $ab \in cl_{\mathcal{A}}(I)$. Thus the result is obtained.

(d). It is clear that $I \subseteq cl_{\mathcal{A}}(I) \cap \mathcal{B}$. We prove the reverse of the inclusion. Suppose that $a \in cl_{\mathcal{A}}(I) \cap \mathcal{B}$. For each $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists $u \in \mathcal{B}$ such that $q_m(a - ua) < \varepsilon/2$. Since $a \in cl_{\mathcal{A}}(I)$, we can find a sequence $(a_n)_{n \in \mathbb{N}}$ in I such that $\lim_{n \rightarrow \infty} a_n = a$, in the topology of \mathcal{A} . It follows that $\lim_{n \rightarrow \infty} p_{\ell}(a_n - a) = 0$, for each $\ell \in \mathbb{N}$. Moreover there exist $\ell_m, n_m \in \mathbb{N}$ and $M_m > 0$ such that

$$q_m(ua_n - ua) = q_m(u(a_n - a)) \leq M_m p_{\ell_m}(a_n - a) q_{n_m}(u).$$

Consequently $\lim_{n \rightarrow \infty} q_m(ua_n - ua) = 0$, for each $m \in \mathbb{N}$. Thus there exists $N_m \in \mathbb{N}$, such that $q_m(ua_{N_m} - ua) < \frac{\varepsilon}{2}$, and so $q_m(ua_{N_m} - a) < \varepsilon$. Since I is closed in \mathcal{B} , thus $a \in I$. This completes the proof. \square

Corollary 3.9. *Let $(\mathcal{B}, q_m)_{m \in \mathbb{N}}$ be a symmetric Segal Fréchet algebra in Fréchet algebra $(\mathcal{A}, p_{\ell})_{\ell \in \mathbb{N}}$ such that $(\mathcal{B}, q_m)_{m \in \mathbb{N}}$ has left approximate units. If I is a closed left ideal in \mathcal{B} , then there exists a closed left ideal J in \mathcal{A} such that $I = J \cap \mathcal{B}$. In fact $J = cl_{\mathcal{A}}(I)$.*

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